ASYMPTOTIC METHODS OF INTEGRATING THE EQUATIONS OF MOTION OF ARTIFICIAL EARTH SATELLITES IN THE PRESENCE OF AERODYNAMIC FORCES

(ASIMPTOTICHESKIE METODY INTEGRIROVANIIA URAVNENII DVIZHENIIA ISKUSSTVENNYKH SPUTNIKOV ZEMLI PRI NALICHII AERODINAMICHESKIKH VOZDEISTVII)

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A large number of papers have been devoted to the problem of orbit evolution for artificia earth satellites subject to aerodynamic forces. The perturbation of satellite motion by the atmosphere is quite complex in character and significantly complicates the investigation of the problem. The basic difficulty is that the action of the atmosphere leads to a slow evolution of the satellite orbit and the numerical integration of the exact equations of motion therefore consumes a large amount of machine time. There thus appears to be a need for either approximate analytical solutions or approximate algorithms for the numerical evaluation of the problem.

A special two-cycle integration method has been proposed for the numerical integration of the satellite's equations of motion [1, 2]. The analytic investigation of the problem can be found in the references [3-5]. A similar review of foreign work on this subject is given in [6].

A number of problems of the motion of a satellite subject to aerodynamic forces and moments is solved in the present paper using the method of averaging [7,8] and the small parameter method. The averaging method is utilized in the form developed by V.M. Volosov [8] for systems with a rapidly rotating phase. The possible application of this method in the solution of problems of satellite dynamics was pointed out by N.N. Moiseev in [9].

Averaging of the equations of motion was carried out using the Laplace method [10], which eliminated the cumbersome computations involved in the usually applied expansions of the averaging functions in series in the eccentricity or Fourier series and enabled a number of different problems connected with the motion of the satellite and its mass center in orbits of high eccentricity to be solved uniquely. A number of new formulas are obtained for determining satellite lifetime. Certain possible trajectories of an aerodynamically controlled satellite are considered. A planar relative motion under the action of a small restoring aerodynamic moment is investigated.

1. Let us write the equations of satellite motion in osculating elements neglecting the non-central gravitational field of the earth and rotation of the atmosphere.

$$\begin{aligned}
\frac{dp}{dt} &= -\frac{r\rho v S}{m}\left[C_x(\alpha)(1+e\cos\vartheta)+eC_y(\alpha)\sin\vartheta\right]\\
\frac{de}{dt} &= -\frac{\rho v S}{2m}\left[2C_x(\alpha)(e+\cos\vartheta)+\frac{(e^2-1)\,C_y(\alpha)\sin\vartheta}{1+e\cos\vartheta}\right]\\
\frac{d\sigma}{dt} &= -\frac{\rho v S}{2me}\left[2C_x(\alpha)\sin\vartheta+\frac{2e+(1+e^2)\cos\vartheta}{1+e\cos\vartheta}C_y(\alpha)\right]\\
\frac{d\vartheta}{dt} &= \frac{\sqrt{\mu p}}{r^2}-\frac{d\sigma}{dt}
\end{aligned} \tag{1.1}$$

Here p is the focal parameter of the osculating conical section, e is the eccentricity, σ is the angular location of the line of apsides, θ is the true anomaly, ρ is the density of the atmosphere, $C_x(\alpha)$, $C_y(\alpha)$ are the drag and lift aerodynamic coefficients dependent on the angle of attack α, S is the characteristic area of the satellite (for example, the mid-section), and m is its mass.

The velocity v and the radius r are determined from the formulas for Keplerian motion

$$v=\sqrt{\frac{\mu}{p}}\sqrt{1+2e\cos\vartheta+e^2},\qquad r=\frac{p}{1+e\cos\vartheta} \tag{1.2}$$

We assume that the density varies exponentially

$$\rho=\rho_1\exp\frac{r_1-r}{H}=\rho_1\rho^\circ \tag{1.3}$$

Here ρ_1 is the density at some characteristic braking altitude r_1, and H is the height of the homogeneous atmosphere.

The atmospheric resistance will be considered to be small. Let us therefore take as the small parameter the quantity

$$\varepsilon=\rho_1 r_1 S\,/\,2m \tag{1.4}$$

which is equal to the aerodynamic overlead at the height r_1 for $e = 0$ and $C_x^2 + C_y^2 = 1$.

Let us introduce a new focal parameter and nondimensional perigee height $p^\circ = p r_1^{-1}$, $r_\pi^\circ = r_\pi \cdot r_1^{-1}$. We must change the independent variable $\tau = t\sqrt{\mu}\, r_1^{-3/2}$. The upper zero index for nondimensional quantities will be omitted in the following. As a result of these transformations the system (1.1) is reduced to the standard form of a system with rapidly rotating phase

$$\frac{dp}{d\tau}=-2\varepsilon\rho\sqrt{p}\left[C_x(\alpha)+\frac{eC_y(\alpha)\sin\vartheta}{1+e\cos\vartheta}\right]\sqrt{1+2e\cos\vartheta+e^2} \tag{1.5}$$

$$\frac{de}{d\tau} = -\frac{\varepsilon\rho}{\sqrt{p}}\left[2(e+\cos\vartheta)C_x(\alpha) + \frac{(e^2-1)C_y(\alpha)\sin\vartheta}{1+e\cos\vartheta}\right]\sqrt{1+2e\cos\vartheta+e^2} \tag{1.6}$$

$$\frac{d\sigma}{d\tau} = -\frac{\varepsilon\rho}{e\sqrt{p}}\left[2\sin\vartheta\, C_x(\alpha) + \frac{[2e+(1+e^2)\cos\vartheta]\,C_y(\alpha)}{1+e\cos\vartheta}\right]\sqrt{1+2\cos\vartheta+e^2} \tag{1.7}$$

$$\frac{d\vartheta}{d\tau} = \frac{(1+e\cos\vartheta)^2}{p^{3/2}} - \frac{d\sigma}{d\tau} \tag{1.8}$$

Letting $\epsilon = 0$, equations (1.5)-(1.8) yield the equation of unperturbed motion

$$\frac{d\vartheta}{d\tau} = \frac{(1+e\cos\vartheta)^2}{p^{3/2}} \qquad (p = \text{const}, \quad e = \text{const}) \tag{1.9}$$

Integrating, we find

$$\tau - \tau_0 = \left(\frac{p}{1-e^2}\right)^{3/2}\left[2\arctan\left(\frac{1-e}{1+e}\right)^{1/2}\tan\frac{\vartheta}{2} - \frac{e\sqrt{1-e^2}\sin\vartheta}{1+e\cos\vartheta}\right] \tag{1.10}$$

For $e < 1$ the unperturbed motion is periodic with a period in τ equal to

$$T = 2\pi p^{3/2}(1-e^2)^{-3/2} \tag{1.11}$$

In order to construct the first approximation we average the right-hand parts of the system (1.5)-(1.7) in τ over a period of the unperturbed motion T. The solution of the equations so derived for slowly varying variables p, e, σ will approximate to the exact solution of the system (1.5)-(1.8) with an error $\sim \epsilon$ in an interval of change $\tau \sim \epsilon^{-1}$. The solution (1.10) for a rapidly varying variable τ has an error ~ 1 in the same interval of change of τ.

For orbits with large eccentricity the system (1.5)-(1.8) can be averaged using the Laplace method [10].

For example, let us evaluate the integral

$$I = \frac{1}{T\rho_2(1-e^2)^{3/2}}\int_0^T \rho\sqrt{1+2e\cos\vartheta+e^2}\,d\tau \tag{1.12}$$

The function $\rho\,(\theta)$ will be given in the form

$$\rho = \rho_2 \exp\frac{\lambda p(\cos\vartheta - 1)}{(1+e)(1+e\cos\vartheta)} \qquad \left(\lambda = \frac{er_1}{H}, \quad \rho_2 = \exp\frac{r_1(1+e-p)}{H(1+e)}\right) \tag{1.13}$$

The true anomaly θ is taken as the variable of integration. Then utilizing (1.9) and (1.10) we get

$$I = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sqrt{1+2e\cos\vartheta+e^2}}{(1+e\cos\vartheta)^2}\exp\left[\frac{\lambda p(\cos\vartheta-1)}{(1+e)(1+e\cos\vartheta)}\right]d\vartheta \tag{1.14}$$

If the orbit eccentricity is sufficiently large ($e > 0.1$), then $\lambda \gg 1$. The nondimensional coefficient λ can be a large positive parameter. (For orbits with the elements $e = 0.2$, $pr_1 = 7842$ km, and $H = 20$ km we have $\lambda = 65.32$).

In the neighborhood of $\theta = 0$ the asymptotic expansion of the integral (1.14) occurs in fractional powers of λ

$$I = \frac{1}{\sqrt{2\pi\lambda p}}\left(1 + \frac{1+3e^2}{8\lambda p}\right) + O(\lambda^{-5/2}) \tag{1.15}$$

The remaining integrals of the system (1.5)-(1.7) are computed analogously. Then for the case α = const we get

$$\frac{dp}{d\tau} = -\frac{2\varepsilon C_x \rho_2 (1-e^2)^{3/2}}{\sqrt{2\pi\lambda}}\left(1+\frac{1+3e^2}{8\lambda p}\right) \tag{1.16}$$

$$\frac{de}{d\tau} = -\frac{2\varepsilon C_x \rho_2 (1+e)(1-e^2)^{3/2}}{p\sqrt{2\pi\lambda}}\left(1+\frac{3e^2-4e-3}{8\lambda p}\right) \tag{1.17}$$

$$\frac{d\sigma}{d\tau} = -\frac{\varepsilon C_y \rho_2 (1+e)(1-e^2)^{3/2}}{pe\sqrt{2\pi\lambda}}\left(1+\frac{3e^2+4e-3}{8\lambda p}\right) \tag{1.18}$$

The right-hand parts of the equations (1.16)-(1.18) exclude terms of the order $\varepsilon\rho_2\lambda^{-5/2}$. To an accuracy of terms of the order $\varepsilon\rho_2\lambda^{-7/2}$ the equation for the nondimensional height of the perigee is obtained as

$$\frac{dr_\pi}{d\tau} = -\frac{\varepsilon\rho_2 C_x (1-e^2)^{3/2}}{\lambda p}\left(\frac{H}{2\pi e r_1}\right)^{1/2}\left(1+3\,\frac{1-4e-e^2}{8\lambda p}\right) \tag{1.19}$$

It follows from (1.19) that if in the system (1.16)-(1.18) terms of order $\varepsilon\rho^2\lambda^{-3/2}$, are excluded, the equations will describe the motion of a satellite with constant perigee height.

Taking eccentricity as the independent variable we get from the equations (1.16)-(1.19)

$$\frac{dr_\pi}{de} = \frac{H}{2e(1+e)r_1}\left(1+H\,\frac{3-4e-3e^2}{4e(1+e)r_\pi r_1}\right) \tag{1.20}$$

$$\frac{d\sigma}{de} = \frac{C_y}{2eC_x}\left(1+\frac{H}{r_\pi r_1(1+e)}\right) \tag{1.21}$$

The system (1.20) and (1.21) contains the small parameter $h = H/r_1$ and therefore its solution can be found by the small parameter method. The solution of (1.20) is sought in the form of the series

$$r_\pi = r_{\pi_0} + h\xi + h^2\eta + \dots \quad (1.22)$$

Substituting (1.22) into (1.20), expanding the right-hand side of the equation in a series in powers of h and equating the coefficients of like powers of h to determine the functions ξ, η we obtain the system

$$\frac{d\xi}{de} = \frac{1}{2e(1+e)}, \qquad \frac{d\eta}{de} = \frac{3-4e-3e^2}{8e^2(1+e)^2 r_{\pi_0}}$$

the solution of which yields the law of variation of the nondimensional height of the perigee

$$r_\pi = r_{\pi_0} + \frac{h}{2}\ln\frac{e(1+e_0)}{e_0(1+e)} + \frac{h^2}{8r_{\pi_0}}\left[10\ln\frac{(1+e)e_0}{(1+e_0)e} + \frac{3+7e_0}{e_0(1+e_0)} - \frac{3+7e}{e(1+e)}\right] \quad (1.23)$$

Solving (1.21) by analogous means we get

$$\sigma = \sigma_0 + \frac{C_y}{2C_x}\ln\frac{e}{e_0} + \frac{hC_y}{2r_{\pi_0}C_x}\ln\frac{(1+e_0)e}{(1+e)e_0} \quad (1.24)$$

The zero index in (1.23) and (1.24) indicates that the value of the functions is given at the initial instant of motion. From (1.23) and (1.24) we find

$$\rho_2 = \rho_0\left[\frac{(1+e)e_0}{(1+e_0)e}\right]^{1/2}\left[1 - \frac{h}{8r_{\pi_0}}\left(10\ln\frac{(1+e)e_0}{(1+e_0)e} + \frac{3+7e_0}{e_0(1+e_0)} - \frac{3+7e}{e(1+e)}\right)\right] \quad (1.25)$$

where ρ_0 is the density at height r_{π_0} referred to the density at the characteristic height r_1.

Substituting (1.25) into (1.16)-(1.18), the averaged system will be evaluated with an error $\sim \varepsilon\lambda^{-5/2}$. Its solution will approximate to the exact solution of (1.1) in the interval $\tau \sim \varepsilon^{-1}$ with an error of the order max $[\varepsilon, \lambda^{-5/2}]$.

The calculation of the duration of motion is reduced to the quadrature

$$\tau = \tau_0 + \frac{r_{\pi_0}}{2\varepsilon\rho_0 C_x}\left(\frac{2\pi(1+e_0)}{he_0}\right)^{1/2}\left[AF(e_0) - AF(e) + \frac{3hJ_1}{4r_{\pi_0}}\right] \quad (1.26)$$

where

$$A = 1 + \frac{3h}{4r_{\pi_0}}\ln\frac{e_0}{1+e_0} + \frac{h}{8r_{\pi_0}}\frac{3+4e_0-3e_0^2}{e_0(1+e_0)}$$

$$F(e) = \frac{3+e}{4(1+e)\sqrt{1-e}} + \frac{1}{8\sqrt{2}}\ln\frac{\sqrt{2}-\sqrt{1-e}}{\sqrt{2}+\sqrt{1-e}}$$

$$J_1 = \int_{e_0}^{e}\ln\frac{e}{1+e}\,\frac{e}{(1+e)^2(1-e)^{3/2}}\,de$$

If terms of order h^2 are neglected in (1.23) and terms of order $\varepsilon^{-1}\sqrt{h}$, are neglected in (1.26), we find the formulas found by P.E. El'iasberg [4]

$$r_\pi = r_{\pi_0} + \frac{h}{2}\ln\frac{(1+e_0)e}{(1+e)e_0}$$

$$\tau = \tau_0 + \frac{r_{\pi_0}}{2\varepsilon C_x \rho_0} \left(\frac{2\pi(1+e_0)}{he_0}\right)^{1/2} [F(e_0) - F(e)]$$

Let us review certain results of calculations based on the approximate formulas (1.23) and (1.26). The initial elements of the satellite orbit are $e_0 = 0.4$, $r_1 p = 9142$ km, and the height of the homogeneous atmosphere $H = 20$ km. The characteristic drag height is assumed as $r_1 = 6520$ km. The aerodynamic characteristics of the satellite are as follows: $C_x = 1$, $C_y = 0$, and $\epsilon = 0.001$.

From the integration of the exact equations of motion it was determined that after 530 revolutions around the earth, the loss of perigee height and the eccentricity of the satellite are $r_1 (r_{\pi_0} - r_\pi) = 13.120$ km, $e = 0.084685$, and the duration of motion $t = 1256.695$ hrs. From the approximate formulas (1.23) and (1.26) for $e = 0.084685$ we have $r_1 (r_{\pi_0} - r_\pi) =$ 13.094 km, and $t = 1256.708$ hrs. In this example the satellite evolution is calculated with great accuracy since here over a large segment of the trajectory the quantity $\lambda^{-5/2}$ is of order ϵ. Consequently, the solution of the approximate system has an error $\sim \epsilon$.

We have assumed an isothermal model of the atmosphere. In practice, for the earth's atmosphere the increase in the drag height reduces the height H of the homogeneous atmosphere. This effect can be taken into account by interpolation of the function $H(r)$ and application of the above method of calculation. The motion of the satellite is affected by a number of factors not yet mentioned: the nonsphericity of the earth, lunar and solar perturbations, etc. All these effects can be represented as an additional small perturbation in the system (1.5)-(1.8) and by application the averaging method.

2. Let us consider the aerodynamic control of the satellite for which the perigee height increases. Such motion can be obtained if at the instant of perigee passage the angle of attack reverses sign, and

$$C_y(\alpha) = \begin{cases} -C_1 < 0 & \text{for } 0 < \vartheta < \pi \\ C_1 > 0 & \text{for } \pi < \vartheta < 2\pi \end{cases} \tag{2.1}$$

Let us substitute the control (2.1) into the system (1.5)-(1.8). The averaged system is of the form

$$\frac{de}{d\tau} = -\frac{\varepsilon\rho_2 (1+e)(1-e^2)^{3/2}}{p\sqrt{2\pi\lambda}} \left[2C_x + \frac{2(1-e)C_1}{\sqrt{2\pi\lambda p}}\right] + O\left(\frac{\varepsilon\rho_2}{\lambda^{3/2}}\right) \tag{2.2}$$

$$\frac{dr_\pi}{d\tau} = -\frac{\varepsilon\rho_2 (1-e^2)^{3/2}}{\sqrt{2\pi\lambda}} \left[\frac{C_x}{\lambda p} - C_1\left(1+\frac{e^2}{\lambda p}\right)\left(\frac{2}{\pi\lambda p}\right)^{1/2}\right] + O\left(\frac{\varepsilon\rho_2}{\lambda^{5/2}}\right) \tag{2.3}$$

$$\frac{d\sigma}{d\tau} = 0 \tag{2.4}$$

It follows from (2.3) that the perigee height will increase only if the lift coefficient is sufficiently large

$$C_1 = |C_y| > C_x \left(\frac{\pi h}{2pe}\right)^{1/2} \left(1 - \frac{eh}{p}\right) \tag{2.5}$$

If the condition (2.5) is fulfilled then the action of the lift force is sufficient to counteract the decrease of perigee height due to drag. With increase in orbit eccentricity, the value of C_y, necessary to increase the perigee height, decreases. The control (2.1) does not alter the location of the line of apsides. From (2.2) and (2.3) we have to within an accuracy of terms $\sim h$

$$\frac{dr_\pi}{de} = -\frac{C_1}{C_x}\left(\frac{hr_\pi}{2\pi e\,(1+e)}\right)^{1/2} + h\,\frac{\pi C_x^2 + (1-e)\,C_y^2}{2\pi e\,(1+e)\,C_x^2} \tag{2.6}$$

The solution is sought in the form of a series in the small parameter h; from (2.2) and (2.6) we get

$$\begin{aligned}\frac{\triangle r}{r_1} = r_\pi - r_{\pi_0} = \frac{2C_1}{C_x}\left(\frac{hr_{\pi_0}}{2\pi}\right)^{1/2} \ln\frac{\sqrt{1+e_0}+\sqrt{e_0}}{\sqrt{1+e}+\sqrt{e}} + \frac{h}{2}\ln\frac{e\,(1+e_0)}{e_0\,(1+e)} + \\ + \frac{C_1^2 h}{2\pi C_x^2}\Big[\ln^2(\sqrt{1+e}+\sqrt{e}) - \ln^2(\sqrt{1+e_0}+\sqrt{e_0}) - \\ - 2\ln(\sqrt{1+e_0}+\sqrt{e_0})\ln\frac{\sqrt{1+e}+\sqrt{e}}{\sqrt{1+e_0}+\sqrt{e_0}} + \ln\frac{e\,(1+e_0)^2}{e_0\,(1+e)^2}\Big]\end{aligned} \tag{2.7}$$

$$\tau = -\frac{r_\pi}{\varepsilon\rho_0 C_x}\left(\frac{h\pi}{2}\right)^{1/2}\int_{e_0}^{e}\frac{\sqrt{e}}{(1-e^2)^{3/2}}\left(\frac{\sqrt{1+e_0}+\sqrt{e_0}}{\sqrt{1+e}+\sqrt{e}}\right)^{\beta} de \qquad \left(\beta = \frac{C_1}{C_x}\left(\frac{2r_{\pi_0}}{\pi h}\right)^{1/2}\right) \tag{2.8}$$

The control (2.1) allows for a drastic change in the satellite lifetime and its orbital elements. We will present some results of calculations. Figure 1 gives the dependence of the satellite time of motion on its orbital eccentricity for the motion with the control (2.1) (curve *1*) and the control

$$C_y(\alpha) = \begin{cases} C_1 > 0 & \text{for} \quad 0 < \vartheta < \pi \\ -C_1 < 0 & \text{for} \quad \pi < \vartheta < 2\pi \end{cases} \tag{2.9}$$

(curve *2*) as well as for the ballistic (uncontrolled) motion (curve *3*). Figure 2 shows how the orbital perigee height varies in these cases. The initial conditions were the same as in the previous example; and it was also assumed that $C_x = |C_y| = 1$. The control (2.9) leads to a lowering of the perigee height and a decrease in the lifetime of the satellite. If for the case of uncontrolled motion the entire duration was 52.3 days, then in the case of (2.9) the satellite existed less than 21 days. By varying the eccentricity from 0.4 to 0.2 the duration of motion with the control (2.1) is 120 days, the perigee height increased by more than 40 km with the result that the satellite attained the orbit where the maximum aerodynamic resistance is more than 10 times less than that in the orbit of an uncontrolled satellite. The approximate formulas (2.7) and (2.8) agree well with the calculation results.

3. *Let us consider the satellite motion under the action of an aerodynamic force directed perpendicularly to the orbit plane. For simplicity of calculation we will assume*

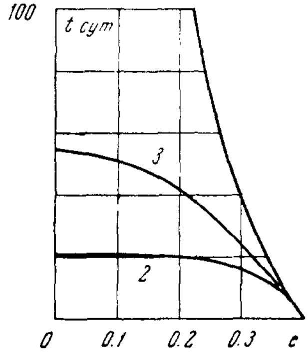

FIG. 1.

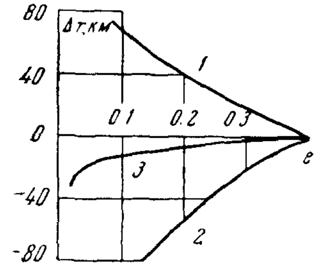

FIG. 2.

$C_y = 0$. To the system (1.5)-(1.8) we add the equations for the new osculating elements

$$\frac{di}{d\tau} = \frac{\varepsilon\rho C_n \cos u\,(1 + 2e\cos\vartheta + e^2)}{\sqrt{p}\,(1 + e\cos\vartheta)}$$
$$\frac{d\Omega}{d\tau} = \frac{\varepsilon\rho C_n \sin u\,(1 + 2e\cos\vartheta + e^2)}{\sqrt{p}\,(1 + e\cos\vartheta)\sin i} \tag{3.1}$$
$$\frac{d\omega}{d\tau} = \frac{d\sigma}{d\tau} - \cos i\,\frac{d\Omega}{d\tau}$$

Here i is the orbit inclination, Ω is the longitude of the ascending node, ω is the angle between the line of apsides and the ascending node, $u = \theta + \omega$ is the argument of the latitude, and C_n is the coefficient of the lateral aerodynamic force. We will compute the system of first approximation for (3.1) to within an accuracy of terms of the order of $\sim\varepsilon\lambda^{-3/2}$

$$\frac{di}{d\tau} = \frac{\varepsilon\rho_2 \cos\omega C_n\,(1 - e^2)^{3/2}}{p\sqrt{2\pi\lambda}}\left[1 + \frac{3e^2 - 8e - 3}{8\lambda p}\right]$$
$$\frac{d\Omega}{d\tau} = \frac{\varepsilon\rho_2 \sin\omega C_n\,(1 - e^2)^{3/2}}{p\sin i\sqrt{2\pi\lambda}}\left[1 + \frac{3e^2 - 8e - 3}{8\lambda p}\right], \qquad \frac{d\omega}{d\tau} = -\cos i\,\frac{d\Omega}{d\tau} \tag{3.2}$$

The first integrals of the system (3.2) are

$$\sin\omega = \frac{\sin i_0 \sin\omega_0}{\sin i}$$
$$\Omega - \Omega_0 = \arcsin\frac{\cos\omega}{\sqrt{1 - \sin^2 i_0 \sin^2\omega_0}} - \arcsin\frac{\cos\omega_0}{\sqrt{1 - \sin^2 i_0 \sin^2\omega_0}} \tag{3.3}$$
$$\arcsin\frac{\cos i}{\sqrt{1 - \sin^2 i_0 \sin^2\omega_0}} - \arcsin\frac{\cos i_0}{\sqrt{1 - \sin^2 i_0 \sin^2\omega_0}} =$$
$$= \frac{C_n}{2C_x}\left[\ln\frac{1 + e}{1 + e_0} + \frac{h\,(e_0 - e)}{2r_{\pi_0}(1 + e)(1 + e_0)}\right]$$

The time of flight is determined by (1.24). The system (3.2) has a particular solution

$$\omega = \Omega = 0, \quad i - i_0 = -\frac{C_n}{2C_x}\left[\ln\frac{1+e}{1+e_0} + \frac{h\,(e_0 - e)}{2r_{\pi_0}\,(1+e)\,(1+e_0)}\right] \tag{3.4}$$

The solution (3.4) describes the basic effect of the lateral force on orbits with large eccentricity: the plane of the orbit rotates about the fixed line of apsides. This result has a simple physical explanation: basic braking occurs at the perigee on highly elongated orbits where the velocity vector is perpendicular to the radius vector of the satellite. The lateral aerodynamic force directed normally to the instantaneous plane of the motion rotates the velocity vector in a plane perpendicular to the radius vector and therefore the line of apsides remains unchanged.

No particular difficulties are encountered in considering more complex manoeuvres of the satellite such as the rotation of the earth's atmosphere.

4. Let us consider the solution of the problem of planar relative motion of the satellite in orbits with high eccentricity. The restoring moment will be assumed small. In this case, the relative motion will be close to a uniform rotation. The analogous problem of the relative motion of the satellite under the action of a small gravitational moment was solved in [11, 12]. The motions close to uniform rotation for a general second order equation were studied by N.N. Moiseev [13]. The motion of the satellite under the action of aerodynamic forces has a number of peculiarities connected with the evolution of the mass center orbit.

We express the equations of motion relative to the mass center as

$$\frac{d^2\varphi}{d\tau^2} + \frac{\varkappa^2\rho f}{p}\sin\alpha\,(1 + 2e\cos\vartheta + e^2) = 0$$
$$\varkappa^2 = \varepsilon b, \qquad b = \frac{r_1 m f F\,\sqrt{S}}{J}, \qquad f = \operatorname{sign} F \tag{4.1}$$

Here J is the cross-sectional moment of inertia, ϕ the angle between the axis of aerodynamic symmetry and a certain fixed direction, and F the coefficient of the restoring moment.

The angle of attack α is expressed through the angles σ, ϕ, and θ according to the formula

$$\alpha = \arcsin\frac{\cos(\sigma - \varphi + \vartheta) + e\cos(\sigma - \varphi)}{\sqrt{1 + 2e\cos\vartheta + e^2}} \tag{4.2}$$

The dependence of the aerodynamic drag coefficient on the angle of attack is approximated by the formula

$$C_x = C_{x_0} - C_1\cos 2\alpha \qquad (C_{x_0} > C_1 > 0)$$

Let us introduce a new variable z by the relation

$$d\varphi \,/\, d\tau = \Omega + \varkappa z \tag{4.3}$$

where Ω is a constant.

Then from (4.1) and (4.3) a system equivalent to the equation (4.1) is obtained

$$\frac{dz}{d\tau} = -\frac{\varkappa\rho f}{p}\sin\alpha\,(1 + 2e\cos\vartheta + e^2), \qquad \frac{d\varphi}{d\tau} = \Omega + \varkappa z \tag{4.4}$$

The system (1.5)-(1.8) and (4.4) describes the relative motion of the satellite and the motion of its mass center. We will estimate the order of magnitude for the quantities in this system. The magnitude of the restoring moment depends on the design parameters of the satellite and the characteristic drag height r_1. If $S/2w = 0.0154$ m^3/kg sec^2, $b = C_x = 1$, then at an altitude of 150 km above the earth's surface $\epsilon = 10^{-4}$ and $\kappa = 10^{-2}$; at 600 km altitude $\epsilon = 2.3 \times 10^{-9}$ and $\kappa = 4.8 \times 10^{-5}$. If the b coefficient is large, say $b \approx 10^6$, then at an altitude of 150 and 600 km the quantity κ is of the order of 10 and 10^{-2} respectively.

Let us first consider the case when

$$\varkappa \ll 1, \qquad \varepsilon \sim \varkappa^2 \tag{4.5}$$

This condition is satisfied if the b coefficient is sufficiently small.

We will use the term "resonance" whenever the frequency of relative motion Ω and the frequency of orbital motion ω are commensurable. It will be shown below that under the condition (4.5) in the non-resonance case in the first approximation in κ, it is possible to consider the relative motion neglecting the measurement of the orbital elements because of the atmospheric action. In the resonance case such a separation is not permissible: the evolution of the orbital motion substantially alters the character of the relative motion.

The system of equations in the first approximation for (4.4) will be ontained by averaging the right-hand parts over T. In the non-resonance case the averaging can be carried out in two independent stages: over ϕ and over θ. The averaging of the first equation of (4.4) yields a zero; in the first approximation the relative motion is a rotation with a constant angular velocity. The aerodynamic moment does not affect the relative motion of the satellite.

We will investigate the resonance case. It is assumed that the difference $\Omega - n\omega s^{-1}$ is a small quantity of order κ where n and s are mutually simple natural numbers. Let us introduce new variables q and the phase γ

$$q = \frac{1}{\varkappa}\left(\frac{n\omega}{s} - \Omega\right), \qquad \gamma = \varphi - \frac{Mn}{s}$$

Here M is the mean anomaly. Within the accuracy of terms of order κ the mean anomaly satisfies the equation

$$dM \,/\, d\tau = \omega \tag{4.6}$$

After a number of transformations and neglecting terms of order κ^2, a standard system follows from the equations (1.5), (1.6), (4.4) and (4.6)

$$\frac{dz}{d\tau} = -\frac{\varkappa\rho f}{p}\sin\alpha\,(1 + 2e\cos\vartheta + e^2), \qquad \frac{d\gamma}{d\tau} = \varkappa\,(z - q)$$
$$\frac{dq}{d\tau} = \frac{3\varkappa\rho C_x\Omega}{b\,(1 - e^2)\,\sqrt{p}}\,(1 + 2e\cos\vartheta + e^2)^{3/2}, \qquad \frac{dM}{d\tau} = \frac{s\Omega}{n} + \varkappa\frac{sq}{n} \tag{4.7}$$

The dependence of M (θ) is found from the Keplerian equations of motion. In the system (4.7) z, q and γ are slowly varying functions, and M is rapidly varying. We will average by the Laplace method the right-hand parts of the system (4.7) over the period of fast motion equal to $2\pi s$ in the variable M. The solution of the averaged system will approximate to the exact solution of (4.7) with an error $\sim \kappa$ in the interval $\tau \sim \kappa^{-1}$. In the case $s = 1, 2$ the averaged system will be

$$\frac{dz}{d\tau} = -\frac{\varkappa\omega f\,(1 + e)\,[1 - (-1)^s]}{2\sqrt{2\pi\lambda}}\cos(\sigma - \gamma), \qquad \frac{dM}{d\tau} = \frac{s}{n}\Omega + \frac{s\varkappa}{n}q$$
$$\frac{dq}{d\tau} = \frac{3\varkappa\Omega\omega\,(1 + e)}{(1 - e)\,b}\left(\frac{p}{2\pi\lambda}\right)^{1/2}[C_{x_0} - C_1\cos(2\sigma - 2\gamma)], \qquad \frac{d\gamma}{d\tau} = \varkappa\,(z - q) \tag{4.8}$$

From (4.8) we find the equation for the angle γ

$$\frac{d^2\gamma}{d\tau^2} + \frac{3\varkappa^2\Omega\omega\,(1 + e)}{b\,(1 - e)}\left(\frac{p}{2\pi\lambda}\right)^{1/2}\times$$
$$\times\left[C_{x_0} - C_1\cos(2\sigma - 2\gamma) + \frac{bf\,[1 - (-1)^s]\,(1 - e)\cos(\sigma - \gamma)}{6\Omega\sqrt{p}}\right] = 0 \tag{4.9}$$

Analogously for the case $s > 2$ we have

$$\frac{d^2\gamma}{d\tau^2} + \frac{3\varkappa^2\Omega\omega\,(1 + e)\,C_{x_0}}{(1 - e)\,b}\left(\frac{p}{2\pi\lambda}\right)^{1/2} = 0 \tag{4.10}$$

The equations (4.9) and (4.10), describe the satellite motion relative to the angle Mn/s. From the system (4.8) we find the expression for the perturbation of the mean anomaly

$$M = M_0 + \frac{s\Omega\tau}{n} + \frac{3\varkappa^2\omega^2\,(1 + e)}{b\,(1 - e)}\left(\frac{p}{2\pi\lambda}\right)^{1/2}\int_0^{\tau}\int_0^{\tau'}[C_{x_0} - C_1\cos(2\sigma - 2\gamma)]\,d\tau\,d\tau' + O(\varkappa) \tag{4.11}$$

Substituting into (4.11) the functions γ (τ) which are the solutions of the equations (4.9) and (4.10) we will determine how the mean anomaly of orbital motion varies with the relative motion of the satellite. In the given approximation the perturbation of the mean anomaly is the only perturbation of the Keplerian motion.

In the case of the main resonance ($s = n = 1$) we get from (4.9) the equation for the determination of the stationary resonance regimes

$$C_{x_0} - C_1\cos(2\sigma - 2\gamma) + C_2\cos(\sigma - \gamma) = 0 \qquad \left(C_2 = \frac{bf\,(1 - e)}{3\Omega\sqrt{p}}\right) \tag{4.12}$$

Solutions of the equations (4.12) correspond to the values of the angle γ for which the angle between the axis of aerodynamic symmetry and mean anomaly remains constant. Such regimes are possible if the modulus of the aerodynamic moment is sufficiently large. The stationary regimes

$$\cos(\sigma - \gamma_0) = \frac{C_2 \pm \sqrt{C_2^2 + 8C_1(C_{x_0} - C_1)}}{4C_1}$$

exist if $C_{x_0} - C_1 < + C_2$. In order to establish stability of the stationary regimes we construct the equation in variations. Its characteristic equation will be

$$k^2 + \frac{3\varkappa^2\Omega^2(1+e)}{b(1-e)}\left(\frac{p}{2\pi\lambda}\right)^{1/2}[C_2 \sin(\sigma - \gamma_0) - 2C_1 \sin(2\sigma - 2\gamma_0)] = 0 \quad (4.13)$$

If the roots of (4.13) are real, then the stationary regimes are unstable. On the phase plane (γ, $d\gamma/d\tau$) they correspond to singular saddle type points. If the roots of (4.13) are purely imaginary, the stationary regimes have focus characteristics. For $s \geqslant 2$ there are no stationary regimes. For sufficiently large values of the b coefficient and the height r_1, the condition $\epsilon \sim 0\ (\kappa^3)$ is satisfied. In this case the first approximation may neglect the perturbation of the satellite mass center motion. The equation of relative motion (4.9) is simplified.

$$\frac{d^2\gamma}{d\tau^2} + \frac{\varkappa^2\omega f(1+e)}{\sqrt{2\pi\lambda}}\cos(\sigma - \gamma) = 0 \quad (4.14)$$

Resonance regimes are possible for $s = 1$ and any n. The equation (4.14) describes the rotational or oscillatory motions relative to the angle Mn. Stationary regimes $\gamma = \sigma + 0.5\pi$ and $\gamma = \sigma + 1.5\pi$ correspond to the uniform rotation of the satellite whereby the axis of aerodynamic symmetry is perpendicular to the line of apsides at the instant of perigee passage. The regime $\gamma = \sigma + 0.5\pi$ is stable if the satellite is statically unstable ($f < 0$). The regime $\gamma = \sigma + 1.5\pi$ is stable for $f > 0$.

The terms of order $\varkappa\lambda^{-3/2}$ were neglected in the averaged system (4.8). A more accurate computation of the right-hand parts in (4.8) is equivalent to the refinement of the coefficients C_{x_0}, C_1 and C_2; qualitatively the character of the relative motion is not altered.

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